Cambridge Assessment Admissions Testing

Sixth Term Examination Paper [STEP]

Mathematics 2 [9470]

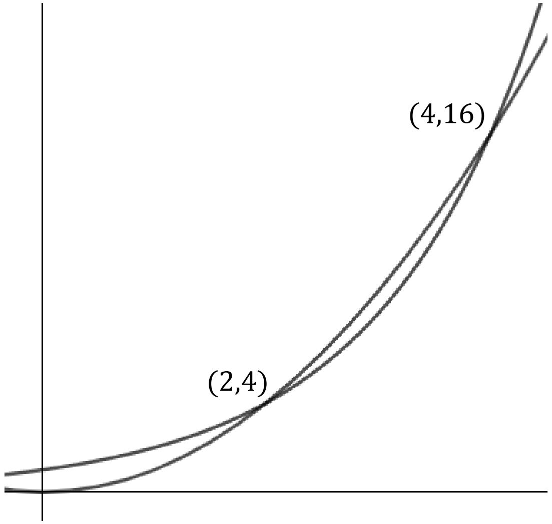
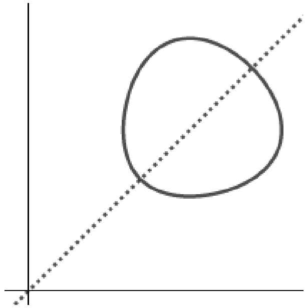
2020

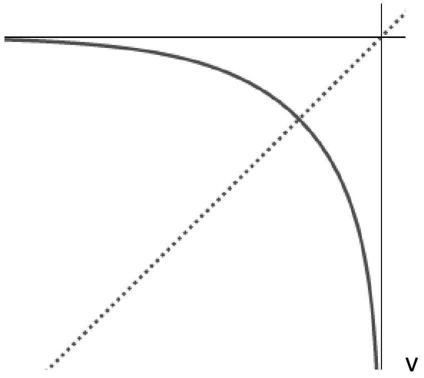
Mark Scheme

			Only penalise missing +c once in parts (i) and (ii)
1(i)	$\int \frac{1}{x^{\frac{3}{2}}(x-1)^{\frac{1}{2}}}\,dx = \int \frac{(1-u)^2}{u^{\frac{1}{2}}}\frac{1}{(1-u)^2}\,du$	**M1** **A1**	Must include attempt at $\frac{du}{dx}$ (or $\frac{dx}{du}$)
	$= 2u^{\frac{1}{2}}$	**A1**	
	$= 2\left(\frac{x-1}{x}\right)^{\frac{1}{2}} + c$	**A1**	
1(ii)	Let $x-2=s$	**M1**	
	Then $\int \frac{1}{(x-2)^{\frac{3}{2}}(x+1)^{\frac{1}{2}}}\,dx = \int \frac{1}{s^{\frac{3}{2}}(s+3)^{\frac{1}{2}}}\,ds$	**A1**	
	Let $s = \frac{3}{u-1}$	**M1** **A1**	
	$\int \frac{1}{s^{\frac{3}{2}}(s+3)^{\frac{1}{2}}}\,ds = \int \frac{(u-1)^2}{3^2 u^{\frac{1}{2}}}\frac{-3}{(u-1)^2}\,du = -\frac{2}{3}u^{\frac{1}{2}}$	**A1**	
	$= -\frac{2}{3}\left(\frac{s+3}{s}\right)^{\frac{1}{2}} = -\frac{2}{3}\left(\frac{x+1}{x-2}\right)^{\frac{1}{2}} + c$	**M1** **A1**	
1(iii)	Let $x = \frac{1+u}{u}$	**M1** **A1**	Any substitution leading to two algebraic factors in the denominator.

	$$\int_2^\infty \frac{1}{(x-1)(x-2)^{\frac{1}{2}}(3x-2)^{\frac{1}{2}}}\,\mathrm{d}x = \int_1^0 \frac{u^2}{(1-u)^{\frac{1}{2}}(3+u)^{\frac{1}{2}}} \cdot \left(\frac{-1}{u^2}\right)\mathrm{d}u$$	**M1** **A1**	If done through a sequence of substitutions: a further substitution leading to a square root of a quadratic as the denominator.
	$$= \int_0^1 \frac{1}{\left(3-2u-u^2\right)^{\frac{1}{2}}}\,\mathrm{d}u$$	**M1**	Expressing the denominator as the square root of a quadratic.
	$$= \int_0^1 \frac{1}{\left(4-(1+u)^2\right)^{\frac{1}{2}}}\,\mathrm{d}u$$	**M1**	
	$$= \left[\arcsin\left(\frac{1+u}{2}\right)\right]_0^1$$	**M1** **A1**	
	$= \frac{1}{2}\pi - \frac{1}{6}\pi = \frac{1}{3}\pi$	**A1**	Answer given.

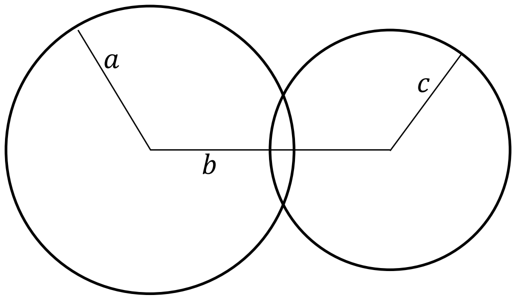
2(i)	$\frac{1-ky}{y}\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{kx-1}{x}$	**M1**	
	$\ln\lvert y\rvert - ky = kx - \ln\lvert x\rvert + c$	**M1** **A1**	
	Hence, $\ln\lvert xy\rvert = k(x+y)+c$	**M1**	
	$xy=\frac{1}{4}\left[(x+y)^2-(x-y)^2\right]=Ae^{k(x+y)}$	**M1** **A1**	
	C_1 is $(x-y)^2=(x+y)^2-2^{x+y}$	**A1**	Answer given.
	C_2 is $(x-y)^2=(x+y)^2-2^{x+y+4}$	**A1**	
	In both cases, the equation is invariant under $(x,y)\mapsto(y,x)$, so symmetrical in $y=x$.	**E1**	
			If the form given for C_1 is differentiated to check that it satisfies the differential equation and it is also checked that it passes through (1,1), award 2 marks.

2(ii)			
	Graphs: Correct shapes of curves	**G1**	
	Graphs: Intersections at (2,4) and (4,16)	**G1**	
	$(x-y)^2 \geq 0$, so $(x+y)^2 > 2^{x+y}$		
	Therefore, $(x+y)$ must lie between 2 and 4	**E1**	Explanation of how the graphs show this needed.
			
	Graph: Symmetry about $y = x$	**G1**	
	Graph: Closed curve lying between $x + y = 3 \pm 1$	**G1**	
	Graph: Passes through (1,1) and (2,2)	**G1**	

2(iii)	Sketches of $y = x^2$ and $y = 2^{x+4}$ $x^2 > 2^{x+4}$ only when $x < -2$.	**M1** **A1**	Marks can be awarded for any justification that $x^2 > 2^{x+4}$ only when $x < -2$.
	 		
	Graph: Symmetry about $y = x$	**G1**	
	Graph: Passes through (-1,-1)	**G1**	
	Graph: $y \to 0$ as $x \to \infty$, $y \to -\infty$ as $x \to 0$	**G1**	

3(i)	Suppose, $\exists k: 2 \leq k \leq n-1$ such that $u_{k-1} \geq u_k$, but $u_k < u_{k+1}$	**M1**	
	Since all of the terms are positive, these imply that $u_k^2 < u_{k-1}u_{k+1}$, so the sequence does not have property *L*.	**E1**	Must include that all terms are positive.
	Therefore, if the sequence has property *L*, once a value k has been reached such that $u_{k-1} \geq u_k$, it must be the case that all subsequent terms also have that property (which is the given definition of unimodality).	**E1**	
3(ii)	$u_r - \alpha u_{r-1} = \alpha\left(u_{r-1} - \alpha u_{r-2}\right)$, so $u_r - \alpha u_{r-1} = \alpha^{r-2}\left(u_2 - \alpha u_1\right)$	**M1** **A1**	 Answer given.
	$u_r^2 - u_{r-1}u_{r+1} = u_r^2 - u_{r-1}\left(2\alpha u_r - \alpha^2 u_{r-1}\right) = \left(u_r - \alpha u_{r-1}\right)^2$ for $r \geqslant 2$	**B1**	Answer given.
	The first identity shows that $u_r > 0$ for all *r* if $u_2 > a u_1 > 0$.	**E1**	
	Since the right hand side of the second identity is always non-negative, the sequence has property *L*, and is hence unimodal.	**E1**	

3(iii)	$u_1 = (2-1)\alpha^{1-1} + 2(1-1)\alpha^{1-2} = 1$, which is correct. $u_2 = (2-2)\alpha^{2-1} + 2(2-1)\alpha^{2-2} = 2$, which is correct.	**M1**	Confirm both cases.
	Suppose that: $u_{k-2} = (4-k)\alpha^{k-3} + 2(k-3)\alpha^{k-4}$, and $u_{k-1} = (3-k)\alpha^{k-2} + 2(k-2)\alpha^{k-3}$.		
	$u_k = 2\alpha\left((3-k)\alpha^{k-2} + 2(k-2)\alpha^{k-3}\right) - \alpha^2((4-k)\alpha^{k-3} + 2(k-3)\alpha^{k-4})$ $= \alpha^{k-1}(6-2k-4+k) + \alpha^{k-2}(4k-8-2k+6)$ $= \alpha^{k-1}(2-k) + 2\alpha^{k-2}(k-1)$ which is the correct expression for u_k	**M1** **A1**	
	Hence, by induction $u_r = (2-r)\alpha^{r-1} + 2(r-1)\alpha^{r-2}$	**E1**	Requires both statement of induction hypotheses and conclusion.
	$u_r - u_{r+1} = \left((2-r)\alpha^{r-1} + 2(r-1)\alpha^{r-2}\right) - \left((1-r)\alpha^r + 2r\alpha^{r-1}\right)$	**M1**	
	$= \alpha^{r-2}\left(2(r-1) + (2-3r)\alpha + (r-1)\alpha^2\right)$	**A1**	
	$= \dfrac{\alpha^{r-2}}{N^2}\left(2N^2(r-1) + (2-3r)N(N-1) + (r-1)(N-1)^2\right)$ $= \dfrac{\alpha^{r-2}}{N^2}\left((r-1) + rN - N^2\right)$	**M1** **A1**	
	when $r = N$, $u_N - u_{N+1} = \dfrac{\alpha^{r-2}(N-1)}{N^2} > 0$	**M1** **A1**	
	when $r = N-1$, $u_{N-1} - u_N = \dfrac{-2\alpha^{r-2}}{N^2} < 0$	**A1**	
	so u_r is largest when $r = N$	**E1**	Answer given.

4(i)	The straight line distance between two points must be less than the length of any other rectilinear path between the points.	**E1**	
4(ii)			
	Diagram showing two circles and straight line joining their centres. Length of line and radii of circles are a, b and c in some order.	**B1**	
	Either statement that the straight line is the longest of the lengths, or explanation that one circle cannot be contained inside the other.	**E1**	
	Explanation that the circles must meet.	**E1**	
4(iii)	**(A)** If $a+b>c$ then $(a+1)+(b+1)>c+2>c+1$ *et cycl.*, so $a+1, b+1, c+1$ can always form the sides of a triangle.	**M1** **A1**	If it is not made clear that the result applies for cyclic permutations, do not award **A1**.
	(B) If $a=b=c=1$ we have $1, 1, 1$ which can form the sides of a triangle.	**B1**	
	If $a=1, b=c=2$ we have $\frac{1}{2}$, 1, 2 which cannot form the sides of a triangle.	**B1**	
	Therefore, $\frac{a}{b}, \frac{b}{c}, \frac{c}{a}$ can sometimes, but not always form the sides of a triangle.	**E1**	
	(C) If $p \geq q \geq r$ then $\|p-q\|+\|q-r\|=p-q+q-r=p-q=\|p-r\|$	**M1** **A1**	
	So two of $\|p-q\|$, $\|q-r\|$, $\|p-r\|$ will always sum to the third, so they never form the sides of a triangle.	**E1**	
	(D) If $a+b>c$ then $a^2+bc+b^2+ca=a^2+b^2-2ab+c(a+b)+2ab$	**M1** **A1**	

	$= (a-b)^2 + c(a+b) + 2ab > c^2 + ab$ *et cycl.* so $a^2 + bc, b^2 + ca, c^2 + ab$ can always form the sides of a triangle.	**M1** **A1**	Penalise not referencing cyclic permutations, but only if not penalised for **(A)**
4(iv)	Since $a + b > a$ *and* $b, \frac{f(a)}{a} > \frac{f(a+b)}{a+b}$ and $\frac{f(b)}{b} > \frac{f(a+b)}{a+b}$	**M1**	
	Since $c < a + b, f(c) < f(a+b)$	**M1**	
	Thus $f(a) + f(b) > \frac{af(a)}{a+b} + \frac{bf(b)}{a+b} = f(a+b) > f(c)$ *et cycl.* So $f(a)$, $f(b)$ and $f(c)$ can form the sides of a triangle.	**M1** **A1**	Penalise not referencing cyclic permutations. Answer given.

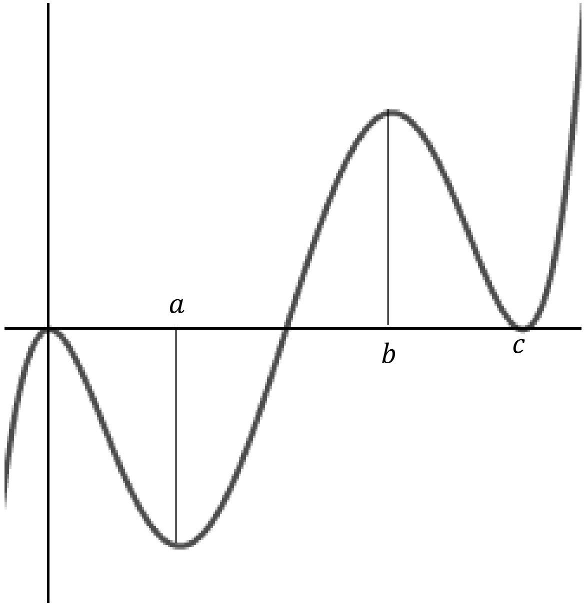
5(i)	$x - q(x) = \sum_{r=0}^{n-1} a_r \times 10^r - \sum_{r=0}^{n-1} a_r = \sum_{r=0}^{n-1} a_r \times (10^r - 1)$	**M1**	
	$10^r \geq 1 \quad \forall r$, so $x - q(x)$ is non-negative	**A1**	Answer given.
	$9 \mid (10^r - 1) \quad \forall r$	**A1**	Answer given.
5(ii)	$x - 44q(x) = 44\big(x - q(x)\big) = 43x$	**M1**	
	So it is a multiple of 9 iff $43x$ is.	**A1**	
	$(43,9) = 1$, so $x - 44q(x)$ is a multiple of 9 iff x is	**A1**	Answer given.
	If x has n digits, $q(x) \leq 9n$	**E1**	
	Since $x = 44q(x)$, $x \leq 396n$. Any n digit number must be at least 10^{n-1}.	**E1**	Answer given
	These inequalities cannot be simultaneously true for $n \geq 5$ $(396 \times 5 < 10^4)$. Therefore $n \leq 4$.	**E1**	Answer given.
	Since $x - 44q(x) = 0$, which is a multiple of 9, x is a multiple of 9.	**M1**	
	$q(x)$ is an integer and $x = 44q(x)$, so x is a multiple of 44. Since $(9,44) = 1$, x must be a multiple of $44 \times 9 = 396$.	**A1**	
	So $x = 396k$ and therefore (by the result above) $k \leq 4$.	**M1**	
	Checking: Only $k = 2$ works.	**A1**	Must be clear that there are no other solutions to get this mark.

5(iii)	$x - 107q\big(q(x)\big) = 0 = 107\big(x - q(x)\big) + 107\left(q(x) - q\big(q(x)\big)\right) - 106x$	M1 A1	
	$(x - q(x))$ and $(q(x) - q\big(q(x)\big))$ are both divisible by 9 (by part (i)) and so x is divisible by 9	A1	
	$x = 107q(q(x))$ and so is divisible by 107, and so is divisible by 963. So $x = 963k$ for some k.	M1	
	If x has n digits, then $q(x) \leq 9n$. By (i), $q\big(q(x)\big) \leq q(x) \leq 9n$. So $x \leq 963n$ and $x \geq 10^{n-1}$ which implies that $n \leq 4$ and so $k \leq 4$	M1 A1	Establishing that $n \leq 4$ Linking to the value of k.
	Checking: Only $k = 1$ works.	A1	

6(i)	Let $\mathbf{M}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$; then $\mathbf{M}^2=\begin{pmatrix} a^2+bc & b(a+d) \\ c(a+d) & d^2+bc \end{pmatrix}$	**M1** **A1**	
	so $\text{Tr}(\mathbf{M}^2) = a^2 + d^2 + 2bc = (a + d)^2 - 2(ad - bc)$	**A1**	Answer given.
6(ii)	Let $\mathbf{M}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$; then $\mathbf{M}^2=\begin{pmatrix} a\tau-\delta & b\tau \\ c\tau & d\tau-\delta \end{pmatrix}$, where $\tau = \text{Tr}(\mathbf{M})$ and $\delta = \text{Det}(\mathbf{M})$.		
	Thus $\mathbf{M}^2 = \pm\mathbf{I} \Leftrightarrow \tau = 0$ and $\delta = \mp 1$ or $b = c = 0$ and $a^2 = d^2 = \pm 1$	**M1** **A1** **A1**	
	If $b = c = 0$ and $a = d = \pm 1$, then $\mathbf{M} = \pm\mathbf{I}$	**E1**	
	If $b = c = 0$ and $a = -d = \pm 1$, then $\tau = 0$ and $\delta = -1$	**E1**	
	Thus $\mathbf{M}^2 = +\mathbf{I} \Leftrightarrow \tau = 0$ and $\delta = -1$.	**E1**	Answer given.
	Thus $\mathbf{M}^2 = -\mathbf{I} \Leftrightarrow \tau = 0$ and $\delta = +1$.	**E1**	Answer given.
6(iii)	Part **(ii)** implies $\text{Det}(\mathbf{M}^2) = -1$, if $\mathbf{M}^4 = \mathbf{I}$, but $\mathbf{M}^2 \neq \pm\mathbf{I}$.	**E1**	
	However, $\text{Det}(\mathbf{M}^2) = \text{Det}(\mathbf{M})^2$, so this is impossible.	**E1**	
	Clearly $\mathbf{M}^2 = \pm\mathbf{I} \Rightarrow \mathbf{M}^4 = \mathbf{I}$	**E1**	
	Part **(ii)** implies that $\mathbf{M}^4 = -\mathbf{I} \Leftrightarrow \text{Tr}(\mathbf{M}^2) = 0$ and $\text{Det}(\mathbf{M}^2) = 1$	**E1**	
	so from **(i)** $\Leftrightarrow \text{Tr}(\mathbf{M})^2 = 2\text{Det}(\mathbf{M})$ and $\text{Det}(\mathbf{M}) = \pm 1$	**M1** **A1**	
	so $\Leftrightarrow \text{Tr}(\mathbf{M}) = \pm\sqrt{2}$ and $\text{Det}(\mathbf{M}) = 1$.	**A1** **A1**	
	Any example, for instance a matrix satisfying the conditions for any of $\mathbf{M}^2 = \mathbf{I}$, $\mathbf{M}^2 = -\mathbf{I}$, $\mathbf{M}^4 = -\mathbf{I}$, which is not a rotation or reflection.	**M1** **A1**	

7(i)	$\|w-1\|^2 = \left\|\frac{1-ti}{1+ti}\right\|^2 = \frac{(1-ti)(1+ti)}{(1+ti)(1-ti)} = 1$, which is independent of t.	**M1** **A1**	
	Points on the line $Re(z) = 3$ have the form $z = 3 + ti$ and the points satisfying $\|w-1\| = 1$ lie on a circle with centre 1.	**E1**	Answer given.
	If $z = p + ti$, then $\|w-c\|^2 = \left\|\frac{2-(p-2)c-cti}{(p-2)+ti}\right\|^2 = \frac{(2-(p-2)c)^2 + c^2t^2}{(p-2)^2+t^2}$	**M1** **M1** **A1**	Choice of form for z.
	which is independent of t when $(2-(p-2)c)^2 = c^2(p-2)^2$	**M1**	
	which is when $c = \frac{1}{p-2}$. Thus the circle has centre at $\frac{1}{p-2}$ and radius $\frac{1}{\|p-2\|}$.	**A1** **A1**	Centre Radius
	$w = \frac{2}{(p-2)+ti} = \frac{2(p-2)-2ti}{(p-2)^2+t^2}$,	**M1**	
	so Im(w) > 0 when $t < 0$; that is, for those z on V with negative imaginary part.	**A1**	

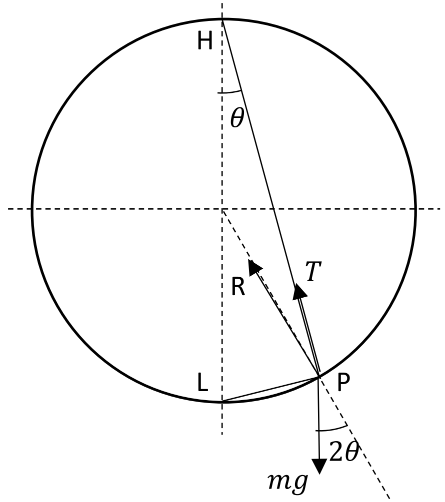
7(ii)	If $z = t + qi$ then $\lvert w - ci\rvert^2 = \left\lvert \dfrac{2 + cq - (t-2)ci}{(t-2)+qi} \right\rvert^2 = \dfrac{c^2(t-2)^2 + (cq+2)^2}{(t-2)^2 + q^2}$	**M1** **M1** **M1** **A1**	Choice of form for z. Recognises the centre is on imaginary axis.
	which is independent of t when $(cq+2)^2 = c^2q^2$	**M1**	
	which is when $c = -\dfrac{1}{q}$ so the circle has centre $-\dfrac{1}{q}i$ **A1** and radius $\sqrt{c^2} = \dfrac{1}{\lvert q\rvert}$ **A1**.	**A1** **A1**	Centre Radius
	$w = \dfrac{2}{(t-2)+qi} = \dfrac{2(t-2) - 2qi}{(t-2)^2 + q^2}$,	**M1**	
	so $\mathrm{Re}(w) > 0$ when $t > 2$; that is, for those z on H with real part greater than 2.	**A1**	

8(i)			
	Graph: Zeroes at $x = 0, c$ and one other point (h: label not required) in (a, b).	**G1**	
	Graph: Turning points at $x = 0, a, b, c$.	**G1**	
	Graph: Quintic shape with curve below axis in $(0, h)$ and above axis in (h, c)	**G1**	
	The area conditions give $F(0) = F(c) = 0$. $F'(x) = f(x)$, so $F'(0) = F'(a) = F'(b) = F'(c) = 0$	**E1**	
	Since f is a quartic and the coefficient of x^4 is 1, F must be a quintic and the coefficient of x^5 is $\frac{1}{5}$. $F(0) = F'(0) = 0$ and $F(c) = F'(c) = 0$, so F must have double roots at $x = 0$ and c. So $F(x)$ must have the given form.	**E1**	Explanation must be clear that the double roots are deduced from the fact that $F(x) = F'(x) = 0$ at those points.
	$F(x) + F(c - x) = \frac{1}{5}x^2(x - c)^2[(x - h) + (c - x - h)]$ $= \frac{1}{5}x^2(c - x)^2(c - 2h)$	**M1** **A1**	

8(ii)	Let A be the (positive) area enclosed by the curve between 0 and a. The maximum turning point of $F(x)$ occurs at $x = b$, with $F(b) = A$. The minimum turning point of $F(x)$ occurs at $x = a$, with $F(a) = -A$.	**E1**	
	Therefore $F(x) \geq -A$, with equality iff $x = a$. So $F(b) + F(x) \geq 0$, with equality iff $x = a$.	**E1**	Answer given.
	$F(a) + F(x) \leq 0$, with equality iff $x = b$.	**B1**	
	Since $F(b) + F(c-b) = \frac{1}{5}b^2(c-b)^2(c-2h)$, either $c > 2h$, or $c = 2h$ and $c - b = a$.	**M1** **A1**	Answer given.
	Also, $F(a) + F(c-a) = \frac{1}{5}a^2(c-a)^2(c-2h)$, so either $c < 2h$, or $c = 2h$ and $c - a = b$.	**M1** **A1**	
	Thus $c = a + b$ and $c = 2h$.	**E1**	Answer given.
8(iii)	$F(x) = \frac{1}{10}x^2(x-c)^2(2x-c)$ So $f(x) = \frac{1}{5}x(x-c)(5x^2 - 5xc + c^2)$	**M1** **A1**	
	The roots of the quadratic factor must be a and b.		
	$f(x) = \frac{1}{5}(5x^4 - 10cx^3 + 6c^2x^2 - c^3x)$ $f'(x) = \frac{1}{5}(20x^3 - 30cx^2 + 12c^2)$ $f''(x) = \frac{1}{5}(60x^2 - 60cx + 12c^2) = \frac{12}{5}(5x^2 - 5cx + c^2)$	**M1** **A1**	
	Therefore $f''(x) = 0$ at $x = a$ and $x = b$ and so $(a, 0)$ and $(b, 0)$ are points of inflection.	**E1**	

9	If the particles collide at time t: $Vt + Ut\cos\theta = d$, and $h - \frac{1}{2}gt^2 = Ut\sin\theta - \frac{1}{2}gt^2$ (or $h = Ut\sin\theta$)	**M1** **M1**	
	Therefore, $d\sin\theta - h\cos\theta = Vt\sin\theta + Ut\sin\theta\cos\theta - Ut\sin\theta\cos\theta$ $= \frac{Vh}{U}$	**M1** **A1**	Answer given.
9(i)	Dividing the previous result by $d\cos\theta$ gives: $\tan\theta - \frac{h}{d} = \frac{Vh}{Ud\cos\theta} > 0$	**M1**	
	Since $\tan\beta = \frac{h}{d}$, $\tan\theta > \tan\beta$ and so $\theta > \beta$	**A1**	Answer given.
9(ii)	The height of collision must be non-negative, so $Ut\sin\theta - \frac{1}{2}gt^2 \geq 0$.	**M1**	
	So $U\sin\theta \geq \frac{1}{2}gt = \frac{1}{2}g\frac{h}{U\sin\theta}$ or $(U\sin\theta)^2 \geq \frac{gh}{2}$ Therefore $U\sin\theta \geq \sqrt{\frac{gh}{2}}$.	**M1** **A1**	Answer given.
9(iii)	$d\sin\theta - h\cos\theta$ can be written as $\sqrt{d^2+h^2}\sin(\theta-\beta)$	**M1** **A1** **A1**	Amplitude β
	So $d\sin\theta - h\cos\theta < \sqrt{d^2+h^2}$ (since $\theta > \beta$)	**A1**	
	Therefore, $\frac{Vh}{U} < \sqrt{d^2+h^2}$ or $\sin\beta = \frac{h}{\sqrt{d^2+h^2}} < \frac{U}{V}$	**M1** **A1**	Answer given.

	The height at which the particles collide is: $h - \frac{1}{2}gt^2 = h - \frac{gh^2}{2U^2 \sin^2\theta}$		
	$h - \frac{gh^2}{2U^2\sin^2\theta} > \frac{1}{2}h$ iff $U^2\sin^2\theta > gh$	**M1** **A1**	
	The vertical velocity of the particle fired from B at the point of collision is: $U\sin\theta - gt = U\sin\theta - \frac{gh}{U\sin\theta}$		
	$U\sin\theta - \frac{gh}{U\sin\theta} > 0$ iff $U^2\sin^2\theta > gh$	**M1** **A1**	
	Since both cases have the same condition: The particles collide at a height greater than $\frac{1}{2}h$ if and only if the particle projected from B is moving upwards at the time of collision.	**E1**	Answer given.

10(i)			
	Diagram showing necessary forces and angles	**G2**	G1 for all forces. G1 for angles used in calculations if not explained well within solution.
	$T = \frac{\lambda(2a\cos\alpha - l)}{l}$	**B1**	
	Resolving tangentially: $T\sin\alpha - mg\sin 2\alpha = 0$	**M1**	
	Therefore $\sin\alpha\left(\frac{\lambda}{l}(2a\cos\alpha - l) - 2mg\cos\alpha\right) = 0$	**A1**	
	Since $\sin\alpha > 0$, $2a\lambda\cos\alpha - \lambda l - 2mgl\cos\alpha = 0$ $\cos\alpha = \frac{\lambda l}{2(a\lambda - mgl)}$	**E1**	
	$\cos\alpha < 1$, so $\lambda l < 2(a\lambda - mgl)$ Therefore $\lambda(2a - l) > 2mgl$	**M1**	
	Since $2a - l > 0$, $\lambda > \frac{2mgl}{2a-l}$	**A1** **E1**	 Explanation that $2a - l > 0$

10(ii)	Energy: $\frac{1}{2}mv^2 - mga\cos 2\theta + \frac{\lambda}{2l}(2a\cos\theta - l)^2 = \frac{1}{2}mu^2 - mga + \frac{\lambda}{2l}(2a - l)^2$	**M1** **A1** **A1**	 Potential Elastic
	If the particle comes to rest when $\theta = \beta$: $-mga\,(2\cos^2\beta - 1) + \frac{\lambda}{2l}(2a\cos\beta - l)^2 = \frac{1}{2}mu^2 - mga + \frac{\lambda}{2l}(2a - l)^2$	**M1**	
	$a\lambda\cos^2\beta\left(\frac{2(a\lambda - mgl)}{\lambda l}\right) - 2a\lambda\cos\beta = \frac{1}{2}mu^2 - 2mga + \frac{2\lambda a^2}{l} - 2a\lambda$	**A1**	
	Therefore, $\cos^2\beta - 2\cos\alpha\cos\beta = \frac{mu^2}{2a\lambda}\cos\alpha + 1 - 2\cos\alpha$	**M1** **A1**	Introduces α
	Adding $\cos^2\alpha$ to both sides: $(\cos\alpha - \cos\beta)^2 = (1 - \cos\alpha)^2 + \frac{mu^2}{2a\lambda}\cos\alpha$	**A1**	Answer given
	For this to occur, $\cos\beta > 0$:	**M1**	
	$\cos^2\alpha > (1 - \cos\alpha)^2 + \frac{mu^2}{2a\lambda}\cos\alpha$	**A1**	
	And so, $u^2 < \frac{2a\lambda}{m}(2 - \sec\alpha)$	**A1**	Answer given

11(i)	If the game has not ended after $2n$ turns, then the sequence has either been n repetitions of HT or n repetitions of TH. So $P(Game\ has\ not\ finished\ after\ 2n\ turns) = 2(pq)^n$. So the probability that the game never ends is $\lim_{n\to\infty} 2(pq)^n = 0$.	**E1**	Answer given.
	Sequence that follows the first H will be k repetitions of TH, followed by H, where $k \geq 0$.	**M1**	
	So $P(A\ wins \mid first\ toss\ is\ H) = \sum_{k=0}^{\infty}(pq)^k p = \frac{p}{1-pq}$	**A1**	Answer given
	$P(A\ wins \cap first\ toss\ is\ H) = p \times \frac{p}{1-pq}$	**M1**	Use of conditional probability
	If first toss is a tail then the sequence that follows would be k repetitions of HT followed by HH.		
	So $P(A\ wins \mid first\ toss\ is\ T) = \frac{p^2}{1-pq}$		
	$P(A\ wins \cap first\ toss\ is\ T) = \frac{p^2 q}{1-pq}$		
	Therefore $P(A\ wins) = \frac{p^2(1+q)}{1-pq}$	**A1**	
11(ii)	Following a first toss of H: A wins with HH or (HT followed by any sequence where A wins after first toss was T) or (T followed by any sequence where A wins after first toss was T)	**E1**	
	The probabilities of these cases are: p^2 $pq\ P(A \text{ wins} \mid \text{the first toss is a tail})$ $q\ P(A \text{ wins} \mid \text{the first toss is a tail})$	**E1**	
	Therefore: $P(\text{A wins} \mid \text{the first toss is a head}) = p^2 + (q + pq)P(\text{A wins} \mid \text{the first toss is a tail})$	**E1**	Answer given.

	Similarly, following first toss of T: A wins with (H followed by any sequence where A wins after first toss was H) or (TH followed by any sequence where A wins after first toss was H)	**M1**	
	Therefore: $P(\text{A wins} \mid \text{the first toss is a tail}) = (p + pq)P(\text{A wins} \mid \text{the first toss is a head})$	**A1**	
	So $P(A \mid H\ first) = p^2 + (q + pq)(p + pq)P(A \mid H\ first)$ $P(A \mid H\ first) = \frac{p^2}{1-(p+pq)(q+pq)}$	**M1**	
	And $P(A \mid T\ first) = (p + pq)(p^2 + (q + pq)P(A \mid T\ first))$ $P(A \mid T\ first) = \frac{p^2(p+pq)}{1-(p+pq)(q+pq)}$	**M1**	
	So $P(A\ wins) = p \times \frac{p^2}{1-(p+pq)(q+pq)} + q \times \frac{p^2(p+pq)}{1-(p+pq)(q+pq)} = \frac{p^2(1-q^3)}{1-(1-p^2)(1-q^2)}$	**A1**	Answer given
11(iii)	Let W be the event that A wins the game. $P(W \mid H\ first) = p^{a-1} + (1 + p + p^2 + \cdots + p^{a-2})qP(W \mid T\ first)$	**M1** **A1**	
	$P(W \mid T\ first) = (1 + q + q^2 + \cdots + q^{b-2})pP(W \mid H\ first)$	**A1**	
	$P(W \mid H\ first) = \frac{p^{a-1}}{1-(1-p^{a-1})(1-q^{b-1})}$	**M1**	
	$P(W \mid T\ first) = \frac{p^{a-1}(1-q^{b-1})}{1-(1-p^{a-1})(1-q^{b-1})}$	**M1**	
	Therefore: $P(W) = \frac{p^{a-1}(1-q^b)}{1-(1-p^{a-1})(1-q^{b-1})}$	**A1**	
	If $a = b = 2$, $P(W) = \frac{p(1-q^2)}{1-(1-p)(1-q)} = \frac{p^2(1+q)}{1-pq}$ as expected.	**B1**	

12(i)	For the biased die: $P(R_1 = R_2) = \sum_{i=1}^{n} \left(\frac{1}{n} + \varepsilon_i\right)^2$	**M1**	
	$P(R_1 = R_2) = \frac{1}{n^2}\sum_{i=1}^{n} 1 + \frac{2}{n}\sum_{i=1}^{n} \varepsilon_i + \sum_{i=1}^{n} {\varepsilon_i}^2$	**M1** **A1**	
	$\sum_{i=1}^{n} \varepsilon_i = 0$, so $P(R_1 = R_2) = \frac{1}{n} + \sum_{i=1}^{n} {\varepsilon_i}^2$	**M1** **A1**	
	For a fair die, $P(R_1 = R_2) = \frac{1}{n}$ and $\sum_{i=1}^{n} {\varepsilon_i}^2 > 0$, so it is more likely with the biased die.	**E1**	
(ii)	$P(R_1 > R_2) = \frac{1}{2}\big(1 - P(R_1 = R_2)\big)$	**M1**	
	Therefore, the value of $P(R_1 > R_2)$ if the die is possibly biased is $\leq P(R_1 > R_2)$ if the die is fair.	**M1**	
	Let T= $\sum_{r=1}^{n} x_r$ and, for each i, let $p_i = \frac{x_i}{T}$ Then $\sum_{i=1}^{n} p_i = 1$, so we can construct a biased n-sided die with $P(X = i) = p_i$	**M1**	
	$P(R_1 > R_2) = \sum_{i=2}^{n} \sum_{j=1}^{i-1} p_i p_j$	**M1** **A1** **A1**	
	For a fair die: $P(R_1 > R_2) = \frac{n-1}{2n}$	**A1**	
	Therefore $\sum_{i=2}^{n} \sum_{j=1}^{i-1} \frac{x_i x_j}{T^2} \leq \frac{n-1}{2n}$ and so $$\sum_{i=2}^{n} \sum_{j=1}^{i-1} x_i x_j \leq \frac{n-1}{2n}\left(\sum_{i=1}^{n} x_i\right)^2$$	**A1**	Answer given.

(iii)	For the biased die: $P(R_1 = R_2 = R_3) = \sum_{i=1}^{n} \left(\frac{1}{n} + \varepsilon_i\right)^3$	**M1**	
	$= \sum_{i=1}^{n} \frac{1}{n^3} + \sum_{i=1}^{n} \frac{3\varepsilon_i}{n^2} + \sum_{i=1}^{n} \frac{3\varepsilon_i^2}{n} + \sum_{i=1}^{n} {\varepsilon_i}^3$	**A1**	
	Therefore $$P(R_1 = R_2 = R_3\ biased) - P(R_1 = R_2 = R_3\ fair) = \sum_{i=1}^{n} \frac{3\varepsilon_i}{n^2} + \sum_{i=1}^{n} \frac{3\varepsilon_i^2}{n} + \sum_{i=1}^{n} {\varepsilon_i}^3$$ $= \sum_{i=1}^{n} \frac{3\varepsilon_i^2}{n} + \sum_{i=1}^{n} {\varepsilon_i}^3$ (since $\sum_{i=1}^{n} \varepsilon_i = 0$)	**A1**	
	$= \sum_{i=1}^{n} \frac{3\varepsilon_i^2}{n} + \varepsilon_i^3 = \sum_{i=1}^{n} \varepsilon_i^2 \left(\frac{3}{n} + \varepsilon_i\right)$		
	But $\varepsilon_i \geq -\frac{1}{n}$ (since $p_i \geq 0$), so this sum must be positive.	**M1** **A1**	
	Therefore, $P(R_1 = R_2 = R_3)$ must be greater for the biased die.	**E1**	

Cambridge Assessment Admissions Testing offers a range of tests to support selection and recruitment for higher education, professional organisations and governments around the world. Underpinned by robust and rigorous research, our assessments include:

- assessments in thinking skills
- admissions tests for medicine and healthcare
- behavioural styles assessment
- subject-specific admissions tests.

We are part of a not-for-profit department of the University of Cambridge.

Cambridge Assessment
Admissions Testing
The Triangle Building
Shaftesbury Road
Cambridge
CB2 8EA
United Kingdom

Admissions tests support:

www.admissionstestingservice.org/help